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AUTHOR(S):

Tachizawa, Kazuya

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# A generalization of the Sobolev-Lieb-Thirring inequality

北大・理・数学 立澤 一哉 (Kazuya Tachizawa)  
Department of Mathematics, Hokkaido University

## 1 Introduction

In this article we explain about a generalization of the Sobolev-Lieb-Thirring inequality and its application. In the proof of our theorem we use the  $\varphi$ -transform of Frazier and Jawerth.

In 1976 Lieb and Thirring proved the following inequality([8]).

**Theorem 1.1 (The Lieb-Thirring inequality)** *Let  $V$  be a non-negative measurable function on  $\mathbb{R}^n$  and*

$$\begin{aligned}\gamma &\geq \frac{1}{2} && \text{for } n = 1, \\ \gamma &> 0 && \text{for } n = 2, \\ \gamma &\geq 0 && \text{for } n \geq 3.\end{aligned}$$

*Then we have*

$$\sum_i |\lambda_i|^\gamma \leq c_{n,\gamma} \int_{\mathbb{R}^n} V^{n/2+\gamma} dx,$$

*where  $\lambda_1 \leq \lambda_2 \leq \dots$  are the negative eigenvalues of the Schrödinger operator  $-\Delta - V$  on  $L^2(\mathbb{R}^n)$ .*

The case  $\gamma > 1/2, n = 1$  or  $\gamma > 0, n \geq 2$  was proved by Lieb and Thirring([8]). The case  $\gamma = 1/2, n = 1$  was proved by Weidl([15]). The case  $\gamma = 0, n \geq 3$  was established by Cwikel([1]), Lieb([7]) and Rozenbljum([9],[10]).

Furthermore Lieb and Thirring proved the following inequality as an application of Theorem 1.1.

**Theorem 1.2 (The Sobolev-Lieb-Thirring inequality)** *Suppose that  $n \in \mathbb{N}$ ,  $\psi_i, |\nabla \psi_i| \in L^2(\mathbb{R}^n)$  ( $i = 1, \dots, N$ ), and that  $\{\psi_i\}_{i=1}^N$  is orthonormal in  $L^2(\mathbb{R}^n)$ . Then we have*

$$\int_{\mathbb{R}^n} \rho^{1+2/n} dx \leq c_n \sum_{i=1}^N \int_{\mathbb{R}^n} |\nabla \psi_i|^2 dx,$$

where

$$\rho(x) = \sum_{i=1}^N |\psi_i(x)|^2.$$

The Sobolev-Lieb-Thirring inequality has important applications such as the stability of matter or the estimates of the dimension of attractors of nonlinear equations (c.f. [5], [8], [14]).

## 2 Proof of Theorem 1.2

In this section we explain about the outline of an alternative proof of Theorem 1.2.

First we recall the definition of  $A_p$ -weights. By a cube in  $\mathbb{R}^n$  we mean a cube which sides are parallel to coordinate axes. A locally integrable function  $w > 0$  a.e. on  $\mathbb{R}^n$  is an  $A_p$ -weight for some  $p \in (1, \infty)$  if there exists a positive constant  $C$  such that

$$\frac{1}{|Q|} \int_Q w(x) dx \left( \frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} \leq C$$

for all cubes  $Q \subset \mathbb{R}^n$ . We say that  $w$  is an  $A_1$ -weight if there exists a positive constant  $C$  such that

$$\frac{1}{|Q|} \int_Q w(y) dy \leq Cw(x) \quad \text{a.e. } x \in Q$$

for all cubes  $Q \subset \mathbb{R}^n$ . We write  $A_p$  for the class of  $A_p$ -weights. It is easy to show  $A_1 \subset A_p$  for  $1 < p < \infty$ . An example of  $A_p$ -weight for  $1 < p < \infty$  is given by  $w(x) = |x|^\alpha \in A_p$  where  $x \in \mathbb{R}^n$  and  $-n < \alpha < n(p-1)$ . Let  $\Omega$  be a bounded  $C^1$ -domain in  $\mathbb{R}^n$ ,  $n \geq 2$ . Then  $w(x) = \text{dist}(x, \partial\Omega)^\alpha$ ,  $(-1 < \alpha < p-1)$ , is another example of  $A_p$ -weight.

For  $f \in L^1_{loc}(\mathbb{R}^n)$ , we define the Hardy-Littlewood maximal operator as

$$M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where the supremum is taken over all cubes  $Q \subset \mathbb{R}^n$  such that  $x \in Q$ . For a nonnegative, locally integrable function  $w$  on  $\mathbb{R}^n$  and  $p \in [1, \infty)$  we set

$$L^p(w) = \{f : \text{measurable}, \|f\|_{L^p(w)} = \left\{ \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right\}^{1/p} < \infty\}.$$

The proof of the following proposition is in [6].

**Proposition 2.1** (i) Let  $1 < p < \infty$  and  $w$  be a nonnegative, locally integrable function on  $\mathbb{R}^n$ . Then  $M$  is bounded on  $L^p(w)$  if and only if  $w \in A_p$ .

(ii) Let  $0 < \tau < 1$ ,  $f \in L^1_{loc}(\mathbb{R}^n)$ , and  $M(f)(x) < \infty$  a.e.. Then  $M(f)(x)^\tau \in A_1$ .

(iii) Let  $1 < p < \infty$  and  $w_1, w_2 \in A_1$ . Then  $w_1 w_2^{1-p} \in A_p$ .

We consider a function  $\varphi$  which satisfies the following properties.

(A1)  $\varphi \in \mathcal{S}(\mathbb{R}^n)$ .

(A2)  $\text{supp } \hat{\varphi} \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\}$ .

(A3)  $|\hat{\varphi}(\xi)| \geq c > 0$  if  $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$ .

(A4)  $\sum_{\nu \in \mathbb{Z}} |\hat{\varphi}(2^{-\nu} \xi)|^2 = 1$  for all  $\xi \neq 0$ .

For  $\nu \in \mathbb{Z}, k \in \mathbb{Z}^n$ ,  $Q = \{(x_1, \dots, x_n) : k_i \leq 2^\nu x_i < k_i + 1, i = 1, \dots, n\}$ , and  $x \in \mathbb{R}^n$ , we set  $\varphi_Q(x) = 2^{\nu n/2} \varphi(2^\nu x - k)$ . The cube  $Q$  described above is called a dyadic cube. Let  $\mathcal{Q}$  be the set of all dyadic cubes in  $\mathbb{R}^n$ .

Now we explain about the outline of a proof of Theorem 1.2. We may assume  $\psi_i \in C_0^\infty(\mathbb{R}^n)$  for  $i = 1, \dots, N$ . Let  $V(x) = \delta \rho(x)^{2/n}$  where  $\delta$  is a positive constant. Then we get  $\int_{\mathbb{R}^n} V^{1+n/2} dx < \infty$ . Set  $v(x) = M(V^\kappa)(x)^{1/\kappa}$ . Then (i) of Proposition 2.1 leads to

$$\int_{\mathbb{R}^n} v^{1+n/2} dx = \int_{\mathbb{R}^n} M(V^\kappa)^{(1+n/2)/\kappa} dx \leq c_1 \int_{\mathbb{R}^n} V^{1+n/2} dx < \infty.$$

Furthermore we have  $v \in A_1$  and  $V \leq v$  a.e..

The following two lemmas are essentially proved by Frazier and Jawerth, where  $(f, g)$  denotes the inner product in  $L^2(\mathbb{R}^n)$  (c.f. [11]).

**Lemma 2.1** There exists an  $\alpha > 0$  such that

$$\alpha \sum_{Q \in \mathcal{Q}} |Q|^{-2/n} |(f, \varphi_Q)|^2 \leq \int_{\mathbb{R}^n} |\nabla f|^2 dx$$

for all  $f \in C_0^\infty(\mathbb{R}^n)$ .

**Lemma 2.2** *Let  $v \in A_2$ . Then there exists a  $\beta > 0$  such that*

$$\int_{\mathbb{R}^n} |f|^2 v \, dx \leq \beta \sum_{Q \in \mathcal{Q}} |(f, \varphi_Q)|^2 \frac{1}{|Q|} \int_Q v \, dx$$

*for all  $f \in C_0^\infty(\mathbb{R}^n)$ .*

By Lemmas 2.1 and 2.2 we have for  $f \in C_0^\infty(\mathbb{R}^n)$

$$\begin{aligned} & \int_{\mathbb{R}^n} |\nabla f|^2 \, dx - \int_{\mathbb{R}^n} V |f|^2 \, dx \\ & \geq \alpha \sum_{Q \in \mathcal{Q}} |Q|^{-2/n} |(f, \varphi_Q)|^2 - \beta \sum_{Q \in \mathcal{Q}} |(f, \varphi_Q)|^2 \frac{1}{|Q|} \int_Q v \, dx. \end{aligned}$$

Let

$$\mathcal{I} = \{Q \in \mathcal{Q} : \beta \int_Q v \, dx > \alpha |Q|^{-2/n}\}$$

and  $\{\mu_k\}_{1 \leq k}$  be the non-decreasing rearrangement of

$$\left\{ \alpha |Q|^{-2/n} - \beta |Q|^{-1} \int_Q v \, dx \right\}_{Q \in \mathcal{I}}.$$

When

$$\mu_k = \alpha |Q|^{-2/n} - \beta |Q|^{-1} \int_Q v \, dx,$$

we define  $\varphi_k = \varphi_Q$ . Then we get

$$\begin{aligned} & \sum_{i=1}^N \int_{\mathbb{R}^n} |\nabla \psi_i|^2 \, dx - \sum_{i=1}^N \int_{\mathbb{R}^n} V |\psi_i|^2 \, dx \\ & \geq \sum_{i=1}^N \sum_{Q \in \mathcal{Q}} |(\psi_i, \varphi_Q)|^2 \left\{ \alpha |Q|^{-2/n} - \beta |Q|^{-1} \int_Q v \, dx \right\} \\ & \geq \sum_{i=1}^N \sum_k \mu_k |(\psi_i, \varphi_k)|^2 = \sum_k \mu_k \sum_{i=1}^N |(\psi_i, \varphi_k)|^2 \\ & = -c \sum_k |\mu_k|. \end{aligned}$$

Now we use the following lemma in [13].

**Lemma 2.3** *There exists a positive constant  $c$  such that*

$$\sum_k |\mu_k| \leq c \int_{\mathbb{R}^n} v^{1+n/2} \, dx,$$

*where  $c$  depends only on  $n$ .*

Hence by Lemma 2.3 we have

$$\begin{aligned} & \sum_{i=1}^N \int_{\mathbb{R}^n} |\nabla \psi_i|^2 dx - \sum_{i=1}^N \int_{\mathbb{R}^n} V |\psi_i|^2 dx \\ & \geq -c \int_{\mathbb{R}^n} V^{1+n/2} dx = -c\delta^{1+n/2} \int_{\mathbb{R}^n} \rho^{1+2/n} dx. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{i=1}^N \int_{\mathbb{R}^n} |\nabla \psi_i|^2 dx & \geq \delta \int_{\mathbb{R}^n} \rho^{1+2/n} dx - c\delta^{1+n/2} \int_{\mathbb{R}^n} \rho^{1+2/n} dx \\ & = \{\delta - c\delta^{1+n/2}\} \int_{\mathbb{R}^n} \rho^{1+2/n} dx. \end{aligned}$$

If we take  $\delta$  small enough, then we get the inequality in Theorem 1.2.

### 3 Some generalizations

We have the following generalization of the Sobolev-Lieb-Thirring inequality for  $n \geq 3$  (c.f. [11, Lemma 3.2], [13]).

**Theorem 3.1** *Let  $n \in \mathbb{N}$ ,  $n \geq 3$ ,  $w \in A_2$  and  $w^{-n/2} \in A_{n/2}$ . Suppose that  $\psi_i \in L^2(\mathbb{R}^n)$ ,  $|\nabla \psi_i| \in L^2(w)$  ( $i = 1, \dots, N$ ), and  $\{\psi_i\}_{i=1}^N$  is orthonormal in  $L^2(\mathbb{R}^n)$ . Then we have*

$$\int_{\mathbb{R}^n} \rho(x)^{1+2/n} w(x) dx \leq c \sum_{i=1}^N \int_{\mathbb{R}^n} |\nabla \psi_i(x)|^2 w(x) dx,$$

where

$$\rho(x) = \sum_{i=1}^N |\psi_i(x)|^2$$

and  $c$  is a positive constant depending only on  $n$  and  $w$ .

An example of  $w$  which satisfies the conditions in Theorem 3.1 is given by  $w(x) = |x|^\alpha$  for  $-n + 2 < \alpha < 2$ .

In the proof of Theorem 3.1 we use the following lemma.

**Lemma 3.1** *Let  $w \in A_2$ . Then there exists an  $\alpha > 0$  such that*

$$\alpha \sum_{Q \in \mathcal{Q}} |Q|^{-2/n} |(f, \varphi_Q)|^2 \frac{1}{|Q|} \int_Q w dx \leq \int_{\mathbb{R}^n} |\nabla f|^2 w dx$$

for all  $f \in C_0^\infty(\mathbb{R}^n)$ .

We omit the detail of the proof of Theorem 3.1.

By Theorem 3.1 we can prove the following  $L^p$  version of the Sobolev-Lieb-Thirring inequality.

**Theorem 3.2** *Let  $n \in \mathbb{N}$ ,  $n \geq 3$  and  $2n/(n+2) < p < n$ . Then there exists a positive constant  $c$  such that for every family  $\{\psi_i\}_{i=1}^N$  in  $L^2(\mathbb{R}^n)$  which is orthonormal and  $|\nabla\psi_i(x)| \in L^p(\mathbb{R}^n)$ , ( $i = 1, \dots, N$ ), we have*

$$\int_{\mathbb{R}^n} \rho(x)^{(1+2/n)p/2} dx \leq c \int_{\mathbb{R}^n} \left( \sum_{i=1}^N |\nabla\psi_i(x)|^2 \right)^{p/2} dx,$$

where

$$\rho(x) = \sum_{i=1}^N |\psi_i(x)|^2$$

and  $c$  depends only on  $n$  and  $p$ .

### Proof

Our proof is very similar to that of the extrapolation theorem in harmonic analysis (c.f. [2, Theorem 7.8]).

Let  $2 < p < n$  and  $2/p + 1/q = 1$ . Let  $u \in L^q$ ,  $u \geq 0$  and  $\|u\|_{L^q} = 1$ . We take a  $\gamma$  such that  $n/(n-2) < \gamma < q$ . Then we have  $u \leq M(u^\gamma)^{1/\gamma}$  a.e and  $M(u^\gamma)^{1/\gamma} \in A_1$ . Furthermore let  $\alpha = \frac{n}{(n-2)\gamma}$ . Then  $0 < \alpha < 1$  and

$$M(u^\gamma)^{-n/(2\gamma)} = \{M(u^\gamma)^\alpha\}^{1-n/2} \in A_{n/2},$$

where we used  $M(u^\gamma)^\alpha \in A_1$  and (iii) of Proposition 2.1. Therefore we have

$$\begin{aligned} \int \rho^{1+2/n} u dx &\leq \int \rho^{1+2/n} M(u^\gamma)^{1/\gamma} dx \leq c \int \left( \sum_{i=1}^N |\nabla\psi_i|^2 \right) M(u^\gamma)^{1/\gamma} dx \\ &\leq c \left( \int \left( \sum_{i=1}^N |\nabla\psi_i|^2 \right)^{p/2} dx \right)^{2/p} \left( \int M(u^\gamma)^{q/\gamma} dx \right)^{1/q} \\ &\leq c \left( \int \left( \sum_{i=1}^N |\nabla\psi_i|^2 \right)^{p/2} dx \right)^{2/p}, \end{aligned}$$

where we used Theorem 3.1 and the inequality

$$\int M(u^\gamma)^{q/\gamma} dx \leq c \int u^q dx = c.$$

If we take the supremum for all  $u \in L^q$ ,  $u \geq 0$  and  $\|u\|_{L^q} = 1$ , then we get

$$\left( \int \rho^{(1+2/n)p/2} dx \right)^{2/p} \leq c \left( \int \left( \sum_{i=1}^N |\nabla \psi_i|^2 \right)^{p/2} dx \right)^{2/p}.$$

Next we consider the case  $2n/(n+2) < p < 2$ . Let

$$f = \left( \sum_{i=1}^N |\nabla \psi_i|^2 \right)^{1/2}.$$

We can take  $\gamma$  such that  $(2-p)n/2 < \gamma < p$ . Then we have

$$M(f^\gamma)^{-(2-p)/\gamma} \in A_2$$

because

$$M(f^\gamma)^{(2-p)/\gamma} \in A_1$$

by (ii) of Proposition 2.1. Furthermore we have

$$\{M(f^\gamma)^{-(2-p)/\gamma}\}^{-n/2} = M(f^\gamma)^{(2-p)n/(2\gamma)} \in A_1 \subset A_{n/2}.$$

Therefore

$$\begin{aligned} \int \rho^{(1+2/n)p/2} dx &= \int \rho^{(1+2/n)p/2} M(f^\gamma)^{-(2-p)p/(2\gamma)} M(f^\gamma)^{(2-p)p/(2\gamma)} dx \\ &\leq \left( \int \rho^{1+2/n} M(f^\gamma)^{-(2-p)/\gamma} dx \right)^{p/2} \left( \int M(f^\gamma)^{p/\gamma} dx \right)^{1-p/2} \\ &\leq c \left( \int f^2 M(f^\gamma)^{-(2-p)/\gamma} dx \right)^{p/2} \left( \int f^p dx \right)^{1-p/2} \\ &\leq c \left( \int M(f^\gamma)^{2/\gamma} M(f^\gamma)^{-(2-p)/\gamma} dx \right)^{p/2} \left( \int f^p dx \right)^{1-p/2} \\ &\leq c \left( \int M(f^\gamma)^{p/\gamma} dx \right)^{p/2} \left( \int f^p dx \right)^{1-p/2} \leq c \int f^p dx, \end{aligned}$$

where we used Theorem 3.1 in the second inequality.

We shall give a generalization of Theorem 3.1. We say a family  $\{\psi_i\}_{i=1}^N \subset L^2(\mathbb{R}^n)$  is suborthonormal if

$$\sum_{i,j=1}^N \xi_i \bar{\xi}_j (\psi_i, \psi_j) \leq \sum_{i=1}^N |\xi_i|^2$$

for all  $\xi_i \in \mathbb{C}$ ,  $i = 1, \dots, N$  (c.f.[5]).



For  $w \in A_2$  and  $s > 0$  let  $\mathcal{H}^s(w)$  be the completion of  $C_0^\infty(\mathbb{R}^n)$  with respect to the norm

$$\|f\|_{\mathcal{H}^s(w)} = \left\{ \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f(x)|^2 w(x) dx + \|f\|^2 \right\}^{1/2}.$$

For any  $Q \in \mathcal{Q}$  there exists a unique  $Q' \in \mathcal{Q}$  such that  $Q \subset Q'$  and the side-length of  $Q'$  is double of that of  $Q$ . We call  $Q'$  the parent of  $Q$ .

We have the following generalization of Theorem 3.1([13]).

**Theorem 3.3** *Let  $n \in \mathbb{N}$ ,  $s > 0$ ,  $\max(1, \frac{n}{2s}) < p \leq 1 + \frac{n}{2s}$ , and  $w \in A_2$ . If  $2s < n$ , then we assume that  $w^{-n/(2s)} \in A_{n/(2s)}$ . If  $2s \geq n$ , then we assume that  $w^{-n/(2s)} \in A_p$  and*

$$\int_{Q'} w dx \leq 2^{2s} \int_Q w dx$$

for all dyadic cubes  $Q$  and its parent  $Q'$ .

Then for  $\{\psi_i\}_{i=1}^N \subset \mathcal{H}^s(w)$  which is suborthonormal in  $L^2(\mathbb{R}^n)$  we have

$$\left\{ \int_{\mathbb{R}^n} \rho(x)^{p/(p-1)} w(x)^{n/(2s(p-1))} dx \right\}^{2s(p-1)/n} \leq c \sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \psi_i(x)|^2 w(x) dx,$$

where

$$\rho(x) = \sum_{i=1}^N |\psi_i(x)|^2$$

and  $c$  is a positive constant depending only on  $n, p, s$ , and  $w$ .

## Remarks

- (1) The case  $s \in \mathbb{N}$  and  $w \equiv 1$  is studied by Ghidaglia, Marion and Temam([5]).
- (2) The case  $w \equiv 1$  is studied by Edmunds and Ilyin([3]) for  $\{\psi_i\}_{i=1}^N$  which is orthonormal in  $L^2(\mathbb{R}^n)$ .
- (3) When  $2s < n$ , an example of  $w$  is given by  $w(x) = |x|^\alpha$  for  $-n + 2s < \alpha < 2s$ .
- (4) When  $2s > n$ , an example of  $w$  is given by  $w(x) = |x|^\alpha$  for  $0 \leq \alpha < \min\{2s - n, n\}$ .
- (5) When  $2s = n$ , our condition means  $w \approx 1$ .

## 4 Estimate of the Hausdorff dimension of the attractor of a nonlinear equation

In this section we apply Theorem 3.1 to a nonlinear equation. In [14] the following result is proved.

**Theorem 4.1** *Let  $\Omega \subset \mathbb{R}^n$  be an open bounded set. Let*

$$g(s) = \sum_{j=0}^{2p-1} b_j s^j, \quad \text{where } b_j \in \mathbb{R}, \quad b_{2p-1} > 0,$$

and

$$\kappa_1 \geq 0, \quad g'(s) \geq -\kappa_1, \quad \forall s \in \mathbb{R}.$$

Let  $d > 0$  and  $u_0 \in L^2(\Omega)$ . Then the equation

$$\begin{cases} \frac{\partial u}{\partial t} - d\Delta u + g(u) = 0 & \text{in } \Omega \times \mathbb{R}_+ \\ u(x, t) = 0 & \text{on } \partial\Omega \times \mathbb{R}_+ \\ u(x, 0) = u_0(x) & x \in \Omega \end{cases}$$

has a unique solution  $u = u(x, t)$  such that

$$u \in L^2(0, T; H_0^1(\Omega)) \cap L^{2p}(0, T; L^{2p}(\Omega)), \quad \forall T > 0$$

and

$$u \in C(\mathbb{R}_+; L^2(\Omega)).$$

Furthermore there exists a maximal attractor  $\mathcal{A}$  which is bounded in  $H_0^1(\Omega)$ , compact and connected in  $L^2(\Omega)$ . Let  $m$  be the integer such that

$$m - 1 < c \left( \frac{\kappa_1}{d} \right)^{n/2} |\Omega| \leq m,$$

where  $c$  is a constant depending only on  $n$ . Then the Hausdorff dimension of  $\mathcal{A}$  is less than or equal to  $m$ .

We have the following result as an application of Theorem 3.1.

**Theorem 4.2** *Let  $n \geq 3$  and  $\Omega \subset \mathbb{R}^n$  be a bounded  $C^2$ -domain. Let*

$$g(s) = \sum_{j=0}^{2p-1} b_j s^j, \quad b_j \in \mathbb{R}, \quad b_{2p-1} > 0$$

and

$$\kappa_1 \geq 0, \quad g'(s) \geq -\kappa_1, \quad \forall s \in \mathbb{R}.$$

Let

$$\begin{aligned} d(x) &= \text{dist}(x, \partial\Omega), \\ -1 + \frac{2}{n} &< a < \frac{2}{n}, \quad w(x) = d(x)^a, \end{aligned}$$

and  $H_0^1(\Omega, w)$  be the completion of  $C_0^\infty(\Omega)$  with respect to the norm

$$\|f\|_{H_0^1(\Omega, w)} = \left\{ \int_{\Omega} (|\nabla f|^2 + |f|^2) w \, dx \right\}^{1/2}.$$

Let  $d > 0$  and  $u_0 \in L^2(\Omega)$ . Then the equation

$$\begin{cases} \frac{\partial u}{\partial t} - d \sum_{i=1}^n \partial_{x_i} (w(x) \partial_{x_i} u) + g(u) = 0 & \text{in } \Omega \times \mathbb{R}_+ \\ u(x, t) = 0 & \text{on } \partial\Omega \times \mathbb{R}_+ \\ u(x, 0) = u_0(x) & x \in \Omega \end{cases}$$

has a unique solution  $u = u(x, t)$  such that

$$u \in L^2(0, T; H_0^1(\Omega, w)), \quad \forall T > 0,$$

and

$$u \in C(\mathbb{R}_+; L^2(\Omega)).$$

Furthermore there exists a maximal attractor  $\mathcal{A}$  which is bounded in  $H_0^1(\Omega, w)$ , compact and connected in  $L^2(\Omega)$ . Let  $m$  be the integer such that

$$m - 1 < c' \left( \frac{\kappa_1}{d} \right)^{n/2} \int_{\Omega} w^{-n/2} \, dx \leq m,$$

Then the Hausdorff dimension of  $\mathcal{A}$  is less than or equal to  $m$ .

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Department of Mathematics  
Faculty of Science, Hokkaido University  
Sapporo 060-0810  
JAPAN  
tachizaw@math.sci.hokudai.ac.jp